

Article

Modeling of Graded-Index Raman Fiber Amplifiers with Pump Depletion

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Abstract: Graded-index (GRIN) fibers are often used for making high-power Raman amplifiers. We employ numerical and semi-analytical techniques to model such amplifiers and include not only the signal's amplification and pump's depletion but also various nonlinear interactions between the signal and pump beams and the self-imaging effects within the GRIN fiber. We solve the coupled nonlinear equations of the pump and signal beams numerically. We also employ the variational technique to obtain simpler equations that can be solved much faster than the full model and still agree with it in most cases of practical interest. We discuss the evolution dynamics of the pump and signal beams, along with a novel process of energy exchange between the two beams because of self-imaging inside the GRIN fiber. The dependence of the signal's amplification on various input parameters is analyzed in detail to optimize the device's design and enhance the signal's amplification for a given pump power and fiber length. Based on our analysis, we establish a *resonant* condition for the maximum energy transfer from the pump to the signal being amplified. We further show that the periodic self-imaging of the pump and signal beams inside a GRIN fiber leads to higher output powers compared to step-index fibers.

Keywords: fiber Raman amplifiers; graded-index fibers; optical fiber amplifiers



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1. Introduction

In recent years, graded-index (GRIN) multimode fibers have been used for making high-power Raman amplifiers [1–4]. With the use of GRIN fibers, the amplified beam's quality [5] is improved considerably at the output of the Raman amplifier due to a phenomenon known as the Raman-induced spatial beam cleanup [6–8]. A distinctive feature of GRIN fibers, known as periodic self-imaging [9], significantly enhances the signal's amplification by creating a nonlinear index grating within the fiber. Recently, self-imaging-based analytical models have been introduced to investigate the mechanisms behind the spatial cleanup of a beam [10,11] and its amplification in GRIN fibers [12]. In a recent study, important nonlinear effects such as self-phase modulation (SPM) and cross-phase modulation (XPM) were included to model a GRIN fiber Raman amplifier with a variational technique [13]. However, the depletion of the pump beam, which often occurs in high-power Raman amplifiers, has not been considered in past studies, with only a few exceptions [14].

In this study, we employ numerical and semi-analytical techniques to model GRIN fiber Raman amplifiers and include not only the pump's depletion but also various nonlinear interactions between the signal and pump beams and the self-imaging effects within the GRIN fiber. We solve the coupled nonlinear equations of the pump and signal beams numerically, but also employ the variational technique to obtain simpler equations that can be solved much faster than the full model and still agree with it in most cases of practical interest. By incorporating the pump's depletion, our analysis provides a more realistic model of the complex dynamics associated with a signal beam being amplified by a stronger

pump beam within a multimode GRIN fiber under Raman amplification. Our variational analysis sheds light on the influence of different relevant processes such as the Raman gain, the SPM and XPM phenomena, and the periodic self-imaging of beams occurring every 1 mm or so within the GRIN fiber amplifier.

In Section 2, we present the full numerical model based on two coupled nonlinear equations for the pump and signal beams. We use these equations in Section 3, with the variational technique, to derive equations for the four parameters associated with each beam, which are assumed to have a Gaussian shape. The variational approach is computationally much more efficient than full numerical simulations but still provides reasonably accurate results. We use the variational equations in Section 4 to obtain simpler coupled equations describing the evolution of the pump and signal powers in a manner unique to GRIN fibers. By taking advantage of this computationally efficient approach, we perform parametric studies that establish the conditions required for maximum amplification in GRIN fibers. We summarize our main results in Section 5 and discuss how they can be used for designing high-power fiber lasers and amplifiers.

2. Theory

We consider a GRIN fiber with a parabolic refractive index profile and include the optical Kerr effect using

$$n(\rho) = n_{core}(1 - \frac{1}{2}b^2\rho^2) + n_2|E|^2, \quad (1)$$

where $\rho = \sqrt{x^2 + y^2}$ is the radial distance from the central axis of the GRIN fiber and n_{core} is the refractive index at $\rho = 0$. Here b , defined as $b = \sqrt{2\Delta}/a$, is related to the index gradient, a is the core's radius, and Δ is the relative difference between the core and cladding index, defined as $\Delta = (n_{core} - n_{clad})/n_{core}$. Figure 1 shows schematics of (a) the GRIN fiber used for Raman amplification and (b) its parabolic refractive index profile.

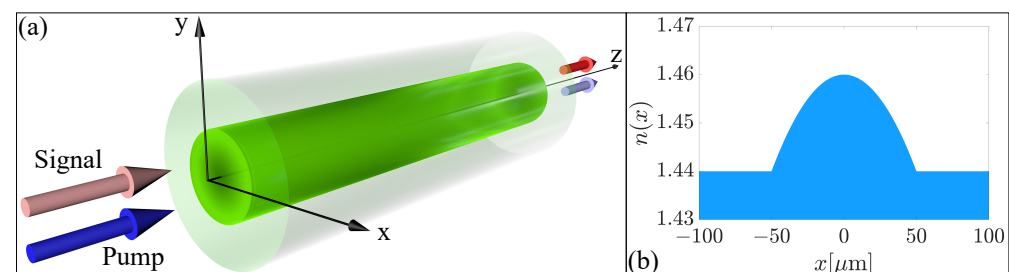


Figure 1. (a) Schematic of a GRIN fiber and (b) its parabolic refractive index profile along one of its transverse axes.

For the silica fibers considered here, the Kerr coefficient n_2 has a value of $2.7 \times 10^{-20} \text{ m}^2/\text{W}$. The silica molecules in the fiber also produce a delayed response in addition to the nearly instantaneous Kerr response of the electrons. This delayed response is known as the Raman response and is accounted for by modifying the Kerr term $n_2|E|^2$ in Equation (1) as follows [15]:

$$n_2|E|^2 = (1 - f_R)n_2|E|^2 + f_R n_2 \int_0^\infty h_R(t') |E(t - t')|^2 dt', \quad (2)$$

where f_R is the fractional Raman contribution (about 18% for silica fibers). The form of the Raman response is quite complex in general but is often approximated using $h_R(t) = (\tau_1^{-2} + \tau_2^{-2})\tau_1 \exp(-t\tau_2) \sin(t/\tau_1)$, with $\tau_1 = 12.2 \text{ fs}$ and $\tau_2 = 22 \text{ fs}$. The response function is normalized as $\int_0^\infty h_R(t) dt = 1$.

When both the pump and signal beams are launched at the input end ($z = 0$) of the GRIN fiber, the total electric field $E(\mathbf{r}, t)$ at a distance z in the fiber can be written as

$$E(\mathbf{r}, t) = \text{Re}\{A_1 \exp[i(k_1 z - \omega_1 t)] + A_2 \exp[i(k_2 z - \omega_2 t)]\}, \quad (3)$$

where $A_j(\rho, z)$ and $k_j = n_{\text{core}}(\omega_j)\omega_j/c$, with $j = 1, 2$, are the amplitudes and wave numbers of the signal and pump beams, respectively. We assume that both waves are polarized in the same direction, and the wavelengths of the pump (λ_1) and signal (λ_2) are 1018 nm and 1060 nm, respectively [2]. For these wavelengths, the frequency shift from the pump to the signal is defined by $\Omega = \omega_1 - \omega_2$. Evaluating the integral in Equation (2) using Equation (3), we obtain

$$\int_0^\infty h_R(t') |E(t - t')|^2 dt' = |A_1|^2 + |A_2|^2 + A_1^* A_2 e^{-i\delta k z} \tilde{h}(\Omega) + A_1 A_2^* e^{i\delta k z} \tilde{h}_R^*(\Omega), \quad (4)$$

where $\delta k = k_1 - k_2$ and $\tilde{h}(\Omega)$ is the Fourier transform of the $h_R(t)$. The Raman gain coefficient g_R is related to the imaginary part of $\tilde{h}_R(\Omega)$ as $g_R = 2f_R n_2(\omega_2/c) \text{Im}(\tilde{h}_R)$. In silica-based GRIN fibers operating at wavelengths near 1 μm , this coefficient is about $g_R = 1 \times 10^{-13} \text{ m/W}$.

By using Equations (2) and (3) in Maxwell's equations and retaining only the phase-matched terms under the slowly varying envelope approximation, we separate the pump and signal terms and obtain the following pair of coupled nonlinear equations for the pump and signal beams [12]:

$$\frac{\partial A_1}{\partial z} + \frac{\nabla_\perp^2 A_1}{2ik_1} + \frac{i}{2} k_1 b^2 \rho^2 A_1 = \frac{i\omega_1}{c} n_2 (|A_1|^2 + 2|A_2|^2) A_1 - \frac{\omega_1}{2\omega_2} g_R |A_2|^2 A_1 \quad (5)$$

$$\frac{\partial A_2}{\partial z} + \frac{\nabla_\perp^2 A_2}{2ik_2} + \frac{i}{2} k_2 b^2 \rho^2 A_2 = \frac{i\omega_2}{c} n_2 (|A_2|^2 + 2|A_1|^2) A_2 + \frac{1}{2} g_R |A_1|^2 A_2, \quad (6)$$

where $\nabla_\perp^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2}$ is the transverse Laplacian. In both equations, the terms containing n_2 describe the nonlinear SPM and XPM effects. The last term containing the Raman gain coefficient g_R is responsible for the amplification of the signal beam and for depletion of the pump beam's power. We can write the two coupled equations in a compact form as

$$\frac{\partial A_l}{\partial z} + \frac{\nabla_\perp^2 A_l}{2ik_l} + \frac{i}{2} k_l b^2 \rho^2 A_l = i \frac{\omega_l}{c} n_2 (|A_l|^2 + 2|A_{3-l}|^2) A_l + (-1)^l \frac{\omega_l}{\omega_2} \frac{g_R}{2} |A_{3-l}|^2 A_l, \quad (7)$$

where $l = 1$ and $l = 2$ correspond to the pump and signal, respectively.

When either the pump or the signal beam propagates inside the GRIN fiber, the contributions of the XPM and Raman terms are absent. Under this condition, the solutions to these equations are known as Gaussian input beams and can be written as [10]

$$A_l(\rho, z) = \sqrt{\frac{I_{l0}}{f_l(z)}} \exp\left(-\frac{\rho^2}{2w_{l0}^2 f_l(z)} + i\Theta(\rho, z)\right), \quad (8)$$

where w_{l0} and I_{l0} are the l th beam's width and peak intensity at $z = 0$ and $f_l(z)$ is a periodic function defined as

$$f_l(z) = \cos^2(bz) + C_l^2 \sin^2(bz), \quad C_l = \frac{\sqrt{1 - P_{l0}/P_{lc}}}{bk_l w_{l0}^2}. \quad (9)$$

Here, $P_{l0} = \pi w_{l0}^2 I_{l0}$ is the input power (in Watts) and $P_{lc} = 2\pi n_{\text{core}}/n_2 k_l^2$ is the critical power at which the self-focusing induces the beam's collapse. For silica fibers, this critical power is about $\approx 6 \text{ MW}$. The function $f_l(\xi)$ describes the effect of GRIN-induced

self-imaging. Both the pump and signal beams compress and expand in a periodic fashion with a period $z_p = \pi/b$, whose numerical value is below 1 mm for most GRIN fibers.

For ease of analysis, we consider the following normalized form of Equation (7):

$$i \frac{\partial \Psi_l}{\partial \xi} + \frac{\delta_l}{2} \left(\frac{\partial^2 \Psi_l}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi_l}{\partial r} \right) - \frac{r^2}{2\delta_l} \Psi_l + \gamma_l |\Psi_l|^2 \Psi_l = -2\Gamma_l |\Psi_{3-l}|^2 \Psi_l + (-1)^l i \frac{G_{Rl}}{2} |\Psi_{3-l}|^2 \Psi_l. \quad (10)$$

We scaled the variables as $\xi = bz$, $r = \rho/w_{20}$, and $\Psi_l = A_l/\sqrt{I_{l0}}$, where ξ and r are normalized distances and Ψ_l is the normalized amplitude. Further, w_{20} is the input width of the signal beam and I_{l0} is the input peak intensity of the l th beam. The four dimensionless parameters in this equation are defined as ($l = 1, 2$):

$$\delta_l = \frac{1}{bk_l w_{20}^2}, \quad \gamma_l = \frac{\omega_l n_2 I_{l0}}{cb}, \quad \Gamma_l = \frac{\omega_l n_2 I_{(3-l)0}}{cb}, \quad G_{Rl} = \frac{g_R \omega_l I_{(3-l)0}}{\omega_2 b}, \quad (11)$$

where γ_l , Γ_l , and G_{Rl} are the normalized SPM, XPM, and Raman loss/gain coefficients for the pump ($l = 1$) and signal ($l = 2$) beams. Using $w_g = (bk_2)^{-1/2}$, where w_g is the width of the fundamental mode of the GRIN fiber, the ratio δ_2 can be written as $\delta_2 = (w_g/w_{20})^2$. In practice, $w_{20} > w_g$, and δ is smaller than 1.

Numerical solutions to Equation (10) provide a reasonably accurate modeling of a GRIN fiber Raman amplifier. As an example, we solve this equation when both the pump and signal beams are Gaussian at the input end, with the widths $w_{10} = 19 \mu\text{m}$ and $w_{20} = 14.3 \mu\text{m}$, respectively, with the standard split-step Fourier transform method [15] in two dimensions, using 1024×1024 in the two transverse dimensions and a step size of 0.001 along the ξ direction. For a preliminary analysis, we neglect the effects of XPM and use a relatively high Raman gain ($G_{R1} = 0.001$ and $G_{R2} = 0.15$) to realize the signal's amplification over a short propagation distance (about 1 cm). The results are shown in Figure 2 with the scaling $\bar{x} = x/w_{20}$ and $\bar{y} = y/w_{20}$.

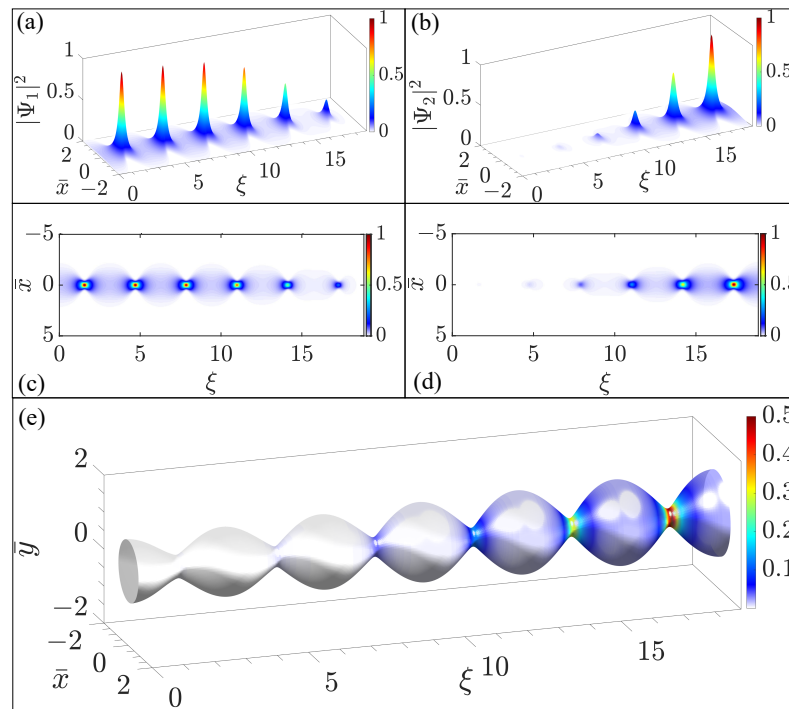


Figure 2. Evolution of the pump's intensity (a) and width (c) and the signal's intensity (b) and width (d) along the amplifier's length (in the plane $\bar{y} = 0$). Periodic patterns are due to self-imaging, which occurs on a millimeter scale inside a GRIN fiber. (e) Isosurface plot of the signal's intensity distribution in three dimensions. Color bars show the intensity scale.

Parts (a) and (b) of Figure 2 shows how the intensity of the pump and signal (in the plane $\bar{y} = 0$) evolve with ξ over a short distance, covering six self-imaging periods. Parts (c) and (d) show how their widths evolve over this distance. As expected, both beams evolve in a periodic fashion owing to self-imaging but their peak intensities change after each period. As the signal beam is amplified, taking energy from the pump beam, its average power increases with distance due to Raman amplification. The three-dimensional evolution pattern of the signal beam is shown in part (e), with colors corresponding to the intensity scale in the color bar. The signal's amplitude gradually increases as energy is transferred from the pump via the Raman amplification process. Along with the amplification in the signal, its width also changes to match the pump's width [13].

3. Variational Analysis

While the numerical solutions of the coupled pump–signal equations in (10) model the Raman amplification process most accurately, obtaining them is computationally time-intensive for realistic values of fiber lengths and various input parameters (more than 10 h for a single set of parameters). Further, numerical simulations provide little insight into what parameters are most relevant for an actual experiment.

To obtain physical insight into the roles of the Raman gain, the SPM and XPM effects, and self-imaging occurring in GRIN fibers, we adopt, in this section, the variational method [16], which can also be applied to nonconservative systems. This method requires a suitable ansatz for the shape of the two beams and further assumes that the perturbations due to the Raman gain and nonlinear effects are relatively small. Both of these limitations are not too serious for GRIN fiber Raman amplifiers, as Gaussian beams are often used in practice, and their gain and nonlinear effects are relatively weak compared to their self-imaging and diffractive effects. However, we still need to check the accuracy of the variational results by comparing them to full numerical simulations, and we do so in this section.

To implement the variational method, we write the coupled pump–signal equations given in Equation (10) in the following form:

$$i \frac{\partial \Psi_l}{\partial \xi} + \frac{\delta_l}{2} \left(\frac{\partial^2 \Psi_l}{\partial r^2} + \frac{1}{r} \frac{\partial \Psi_l}{\partial r} \right) - \frac{1}{2\delta_l} r^2 \Psi_l + \gamma_l |\Psi_l|^2 \Psi_l = i\epsilon_l, \quad (12)$$

where we treat the XPM and Raman terms as being small and lump them into a perturbation ϵ_l , defined as

$$\epsilon_l = \left(2i\Gamma_l + (-1)^l \frac{G_{Rl}}{2} \right) |\Psi_{3-l}|^2 \Psi_l. \quad (13)$$

The Lagrangian densities corresponding to Equation (12) are given by [10,13]

$$\mathcal{L}_l = \frac{i}{2} r (\Psi_l \partial_\xi \Psi_l^* - \Psi_l^* \partial_\xi \Psi_l) + \frac{\delta_l}{2} r |\partial_r \Psi_l|^2 - \frac{\gamma_l}{2} r |\Psi_l|^4 + \frac{r^3}{2\delta_l} |\Psi_l|^2 + ir(\epsilon_l \Psi_l^* - \epsilon_l^* \Psi_l), \quad (14)$$

where $\partial_j = \partial/\partial j$ for $j = \xi, r$.

In the absence of any perturbation, the solution to Equation (12) is given in Equation (8) for initially Gaussian-shaped pump and signal beams. With perturbation included, we assume that the solution remains in the form of a chirped Gaussian beam for both the pump ($l = 1$) and signal ($l = 2$). We include the parabolic curvature of the phase front as

$$\Psi_l(r, \xi) = \psi_l(\xi) \exp \left[-\frac{r^2}{2r_l^2(\xi)} + id_l(\xi)r^2 + i\phi_l(\xi) \right], \quad (15)$$

where ψ_l , r_l , d_l , and ϕ_l are four parameters representing the amplitude, width, phase front curvature, and phase of the beams, respectively. As a result of the perturbation, these parameters vary with the propagation of the beams inside the GRIN fiber and become functions of ξ .

At this point, we follow the standard Rayleigh–Ritz optimization process [17] and obtain the reduced Lagrangian for both beams by integrating the corresponding Lagrangian densities over r as $L_l = \int_0^\infty \mathcal{L}_l dr$:

$$L_l = \frac{1}{2} \psi_l^2 r_l^2 \left(\frac{d\phi_l}{d\xi} \right) + \left(2\delta_l d_l^2 + \frac{1}{2\delta_l} + \frac{dd_l}{d\xi} \right) \frac{\psi_l^2 r_l^4}{2} + \frac{\delta_l}{4} \psi_l^2 - \frac{\gamma_l}{8} \psi_l^4 r_l^2 + i \int_0^\infty r (\epsilon \Psi_l^* - \epsilon^* \Psi_l) dr. \quad (16)$$

By employing the *Euler–Lagrange* equation, $\partial_\xi (\partial_{X_\xi} L_l) - \partial_X L_l = 0$, using the eight parameters given by $X_l = \psi_l, r_l, d_l, \phi_l$ ($l = 1, 2$), we obtain eight coupled ordinary differential equations (ODEs) that describe the evolution of the parameters of each beam along the fiber’s length:

$$\frac{d\psi_l}{d\xi} = -2\delta_l d_l \psi_l + (-1)^l \frac{G_{RI}}{2} \psi_{3-l}^2 \left(\frac{r_e}{r_l} \right)^2 \left[2 - \left(\frac{r_e}{r_l} \right)^2 \right] \psi_l \quad (17a)$$

$$\frac{dr_l}{d\xi} = 2\delta_l d_l r_l - (-1)^l \frac{G_{RI}}{2} \psi_{3-l}^2 \left(\frac{r_e}{r_l} \right)^2 \left[1 - \left(\frac{r_e}{r_l} \right)^2 \right] r_l \quad (17b)$$

$$\frac{dd_l}{d\xi} = -2\delta_l d_l^2 - \frac{\gamma_l}{4} \left(\frac{\psi_l}{r_l} \right)^2 - \frac{1}{2\delta_l} + \frac{\delta_l}{2r_l^4} - 2\Gamma_l \psi_{3-l}^2 \left(\frac{r_e}{r_l} \right)^2 \left[1 - \left(\frac{r_e}{r_l} \right)^2 \right] \quad (17c)$$

$$\frac{d\phi_l}{d\xi} = -\frac{\delta_l}{r_l^2} + \frac{3}{4} \gamma_l \psi_l^2 + 2\Gamma_l \psi_{3-l}^2 \left(\frac{r_e}{r_l} \right)^2 \left[2 - \left(\frac{r_e}{r_l} \right)^2 \right], \quad (17d)$$

where $1/r_e^2 = 1/r_1^2 + 1/r_2^2$ and $l = 1, 2$. In the absence of nonlinear effects and Raman gain, there exists a stable fixed point at $r_l = \sqrt{\delta_l}$ for both beams. It corresponds to the fundamental mode of the GRIN fiber at the pump and signal wavelengths.

The numerical solutions of the preceding eight coupled ODEs are obtained much faster than a full numerical simulation of Equation (10). We solve them using the fourth-order Runge–Kutta method and compare the variational results in Figure 3 to the numerical solutions of Equation (10). A remarkably good agreement between the two sets of results is clearly seen in this figure. The values of the pump/signal parameters and the normalized coefficients used are given in Table 1. As the signal is amplified, its width tends towards the pump’s width; a feature seen in a previous study without pump depletion included [13]. A relatively short distance (about 1 cm) was used for the simulations in Figure 3 for resolving rapid oscillations in the widths and intensities of the pump and signal beams resulting from self-imaging. These oscillations follow Equation (9) over a short distance. The question remains as to what happens in the longer fibers (>1 m) used in practice for Raman amplifiers. To answer this question and to provide a complete picture of the amplification process, we consider longer distances in the next section.

Table 1. List of parameters used for Figure 3 and their values.

Parameters	Symbols	Values [Units]
Input Pump Power	P_{10}	114 [kW]
Input Signal Power	P_{20}	1.14 [kW]
Input Pump Width	w_{10}	19 [μm]
Input Signal Width	w_{20}	14.34 [μm]
	δ_1	0.19
	δ_2	0.2
Raman Pump Loss	G_{R1}	6.5×10^{-5}
Raman Signal Gain	G_{R2}	0.0035
Pump XPM	Γ_1	1.04×10^{-4}
Signal XPM	Γ_2	0.0057
Pump SPM	γ_1	0.0059
Signal SPM	γ_2	1×10^{-4}

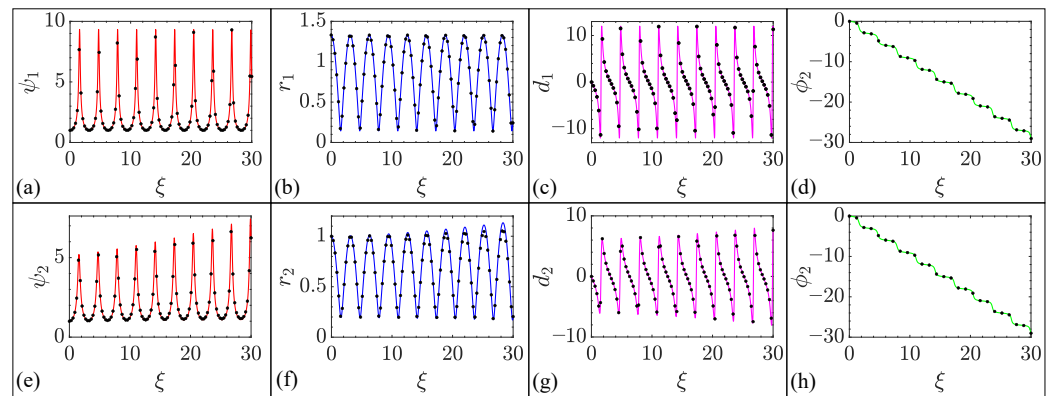


Figure 3. Comparison of variation results (solid lines) with numerical simulations (black dots) for $w_{10} = 1.33w_{20}$. The evolution of the four parameters (ψ_l, r_l, d_l, ϕ_l) of the pump (a–d) and signal (e–h) beams with distance is shown inside a GRIN fiber amplifier.

4. Results and Discussion

Having verified that the variational analysis provides reasonably accurate results, we use it to study how various input parameters can be adjusted to improve the performance of a GRIN fiber Raman amplifier. From the coupled ODEs in Equation (17), we note that the parameters of both beams are mutually influenced through XPM and the Raman gain. However, self-imaging leads to rapid oscillations in all beam parameters, a feature of GRIN fibers that masks the long-range evolution of these parameters. For better visualization, we filter out the self-imaging oscillations by averaging over one self-imaging period ($\xi_p = \pi$) along the propagation direction. While $\langle d_l \rangle$ remains around zero for both the pump and signal beams, $\langle \psi_2 \rangle$ for the signal steadily increases because of Raman amplification, while $\langle \psi_1 \rangle$ decreases because of the pump's depletion. Moreover, the average widths of both beams change and display unusual behavior under the effects of XPM and the Raman gain. We focus on these aspects in what follows.

4.1. Effect of Raman Gain and XPM on Beam Widths

The evolution of the average widths ($\langle r_1 \rangle$ and $\langle r_2 \rangle$) of the pump and signal beams along the fiber's length is shown in Figure 4, for three different initial pump widths and a fixed signal width ($w_{20} = 14.3 \mu\text{m}$), until the pump is depleted to 0.1% of its initial power. The three cases correspond to (i) a pump narrower than the signal ($w_{10}/w_{20} = 0.67$), (ii) the same width for both beams ($w_{10}/w_{20} = 1$), and (iii) a pump wider than signal ($w_{10}/w_{20} = 1.33$). The variational results (solid lines) are compared with the full numerical analysis (dashed lines) in all cases. The parameter values used for this figure are given in Appendix A.

When the effects of XPM are ignored (red lines), both widths change in a monotonic fashion, without any oscillations, such that $\langle r_2 \rangle$ evolves asymptotically towards the initial pump width. The pump's width $\langle r_1 \rangle$ decreases with distance for $w_{10} < w_{20}$ and exhibits an opposite trend in the other two cases. When the effects of XPM are included (black lines), $\langle r_2 \rangle$ oscillates about the initial pump width and eventually settles near it. Because of XPM, $\langle r_1 \rangle$ exhibits a forced oscillatory behavior about its initial value. The amplitude of the oscillations increases the width distance as the pump's power decreases because of its depletion. The variational and numerical results mostly agree, but differences occur for $\langle r_1 \rangle$ after a significant depletion of the pump. This is probably due to the large changes in the phase of the pump beam that occur through the signal-induced XPM. The SPM effects are included but play a minor role in Figure 4. This is because the power levels for both the pump and signal remain well below the critical level of self-focusing ($P_l \ll P_c$). Moreover, SPM-induced beam compression, observed in the absence of pump depletion [13], does not occur. It is the Raman effect that leads to spatial cleanup through the narrowing of the

signal beam. This is further corroborated by the SPM and Raman coefficients given in the tables in Appendix A ($\gamma_2 \ll G_{R2}$).

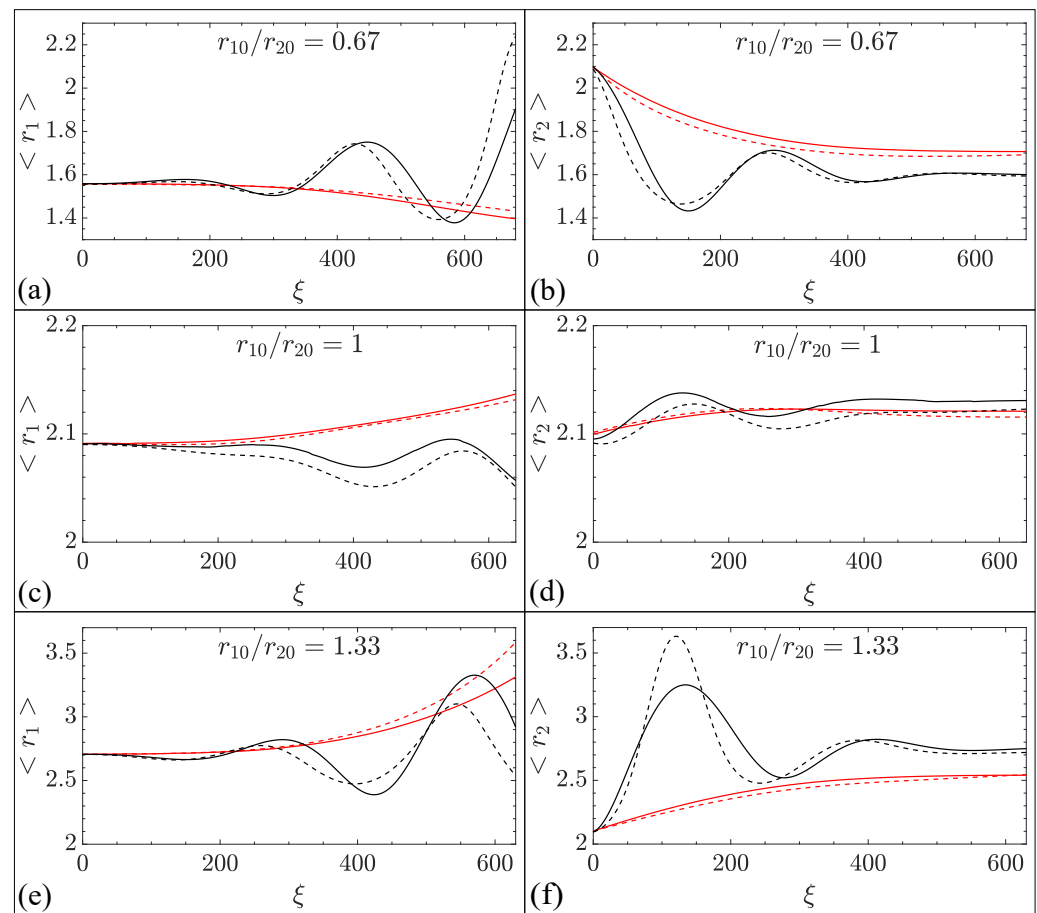


Figure 4. Impact of the pump’s initial width on the evolution of average beam widths for a given initial signal beam width ($w_{20} = 14.3 \mu\text{m}$). The average values of $\langle r_1 \rangle$ (left) and $\langle r_2 \rangle$ (right) are plotted as a function of ξ for narrower (a,b) and broader (e,f) pump pulses. The middle row (c,d) shows the equal-width case. XPM is neglected for red curves. The variational results (solid lines) are compared with numerical ones (dashed lines) in all cases.

4.2. Impact of Pump Depletion

For any amplifier, the most relevant issue is how much the signal is amplified for a given set of parameters and what role the pump’s depletion plays. It is thus important to consider the evolution of the signal and pump powers along the amplifier’s length. The powers for both beams are obtained by integrating over each beam’s cross-section: $\bar{P}_l = \iint |\Psi_l|^2 da$ [18]. Owing to their cylindrical symmetry and Gaussian shapes, the beam powers at any distance are obtained using $\bar{P}_l = \pi \psi_l^2 r_l^2$. Using Equations (17a) and (17b), powers are found to evolve as

$$\frac{d\bar{P}_l}{d\xi} = (-1)^l \pi G_{Rl} \psi_1^2 \psi_2^2 \left(\frac{r_1^2 r_2^2}{r_1^2 + r_2^2} \right) = (-1)^l \frac{G_{Rl}}{\pi} \frac{\bar{P}_1 \bar{P}_2}{r_1^2 + r_2^2}, \quad (18)$$

where $l = 1$ for the pump and $l = 2$ for the signal.

When the two beams are launched with their widths matched to the corresponding widths of the fundamental mode of the GRIN fiber, GRIN-induced self-imaging does not

occur, as only a single mode of the fiber is excited by both beams. Under this condition, Equation (18) reduces to the simple form

$$\frac{d\bar{P}_l}{d\xi} = (-1)^l G_{Rl}^0 \bar{P}_1 \bar{P}_2, \quad G_{Rl}^0 = \frac{G_{Rl}}{[\pi(r_{10}^2 + r_{20}^2)]}, \quad (19)$$

where r_{10} and r_{20} are the initial beam widths. This expression agrees with the well-known result for single-mode waveguides [18].

For the three cases in Figure 4, the evolution of the pump and signal powers along the fiber's length under the combined effects of XPM and the Raman gain is shown in Figure 5. A remarkable feature in all cases is that the beam's power increases for the signal and decreases for the pump in a stepwise fashion. This feature is unique to GRIN fibers as it is related to the self-imaging phenomenon. The Raman amplifier's gain depends on the local pump intensity, which is enhanced periodically because of GRIN-induced focusing on a millimeter-length scale. The step-like changes seen in Figure 5 take place at these locations, where r_1 and r_2 become minimum, and the effective gain/loss coefficient, given by $G_{Rl}/(r_1^2 + r_2^2)$, becomes maximum.

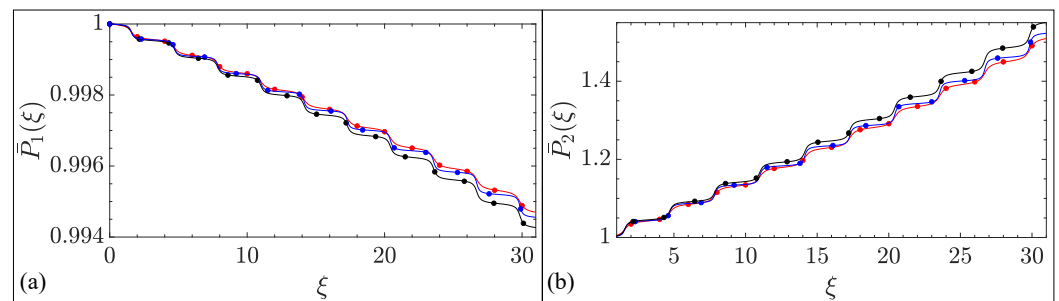


Figure 5. Evolution of the (a) signal and (b) pump powers in the three cases shown in Figure 4: $w_{10}/w_{20} = 0.67$ (red), 1 (black), and 1.33 (blue). In all cases, the variational results (solid line) are compared with the numerical ones (solid dots).

4.3. Impact of Pump Beam's Width

A salient feature seen in Figure 5 is that the pump and signal powers at any distance within the GRIN fiber also depend on the initial width of the pump beam relative to the signal's width. Even though the changes are relatively small over the short distance used for Figure 5, they become significant over the typical lengths (10 m or more) used for Raman amplifiers.

To understand the dependence of signal power on the input pump width, we vary w_{10} while keeping the signal's initial width (w_{20}) fixed, as well as the pump power. By repeating this process for three values of w_{20} , we calculate the amplifier gain for the signal as the ratio of the output to input powers ($G_2 = P_{out}/P_{in}$) for a GRIN fiber of fixed length. The same ratio for the pump indicates the extent of its depletion.

Both of these ratios are plotted in Figure 6 as a function of the input pump's width for input signal beams of widths 10 μm , 15 μm , and 20 μm . The initial pump and signal powers are $P_{10} = 100$ kW and $P_{20} = 0.1$ kW, respectively. The fiber length corresponds to 128 self-imaging periods (≈ 14 cm) and is chosen such that a significant amplification of the signal takes place without the complete depletion of the pump. A surprising feature seen in Figure 6 is that the maximum amplification occurs for a specific value of w_{10} , which changes with w_{20} . In fact, the maximum gain occurs when the initial widths of the pump and signal beams are the same ($w_{10} = w_{20}$).

Consider the effects of the pump parameters on the signal's amplification. The normalized gain coefficient for the signal, $G_{R2} = g_R P_{10}/(\pi b w_{10}^2)$, has an inverse square dependence on the pump's width. This relationship suggests that, for a fixed pump power, an increase in the pump's width should decrease the gain, at least initially. However, as we have seen in Figure 3, the pump's width does not remain constant in a GRIN fiber

and exhibits periodic contraction and expansion on a millimeter scale due to self-imaging. Moreover, as seen in Figure 4, even its average value oscillates on a longer-length scale because of XPM effects. As the widths of both the pump and signal beam oscillate with distance in a synchronized *resonant* fashion, the maximum transfer of energy between them occurs when the two beam widths are the same initially.

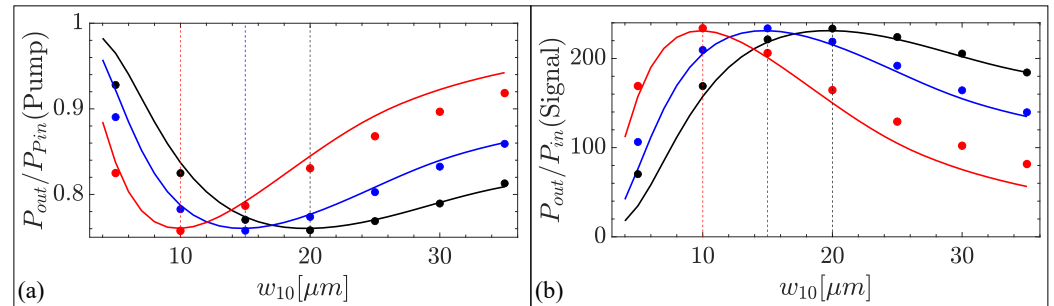


Figure 6. The output powers of the pump (a) and signal (b) beams plotted as a function of the initial pump width for fixed signal widths of 10 μm (red), 15 μm (blue), and 20 μm (black). The variational results (lines) are in good agreement with the numerical results (dots).

4.4. GRIN Advantage

An important question is whether the use of GRIN fibers creates better Raman amplifiers. One well-known advantage is their Raman-induced beam cleanup, which leads to an improvement in the spatial quality of the amplified beam, as judged by the so-called M^2 factor. Here, we show that GRIN-induced self-imaging leads to another advantage by providing higher output powers.

To show this clearly, we compare, in Figure 7, the evolution of (a) pump and (b) signal powers along the fiber's length with and without self-imaging effects. The pump and signal beams have the same input widths but their powers differ by a factor of 1000 ($P_{10} = 100$ kW and $P_{20} = 0.1$ kW). At any distance inside the fiber, the signal's power is larger (and the pump power is lower) when self-imaging is included. The reason for this is related to the periodic focusing of the pump beam, which enhances its intensity by a large factor and increases the amplifier's gain by the same factor in the focusing region. To quantify the GRIN advantage, we consider the gain per unit length in the case presented in Figure 7. This metric is 30% larger in the case of GRIN fibers compared to step-index fibers, for which self-imaging is absent. The periodic increase in gain is shown in parts (c) and (d), where the effective loss/gain coefficients are plotted over two self-imaging periods.

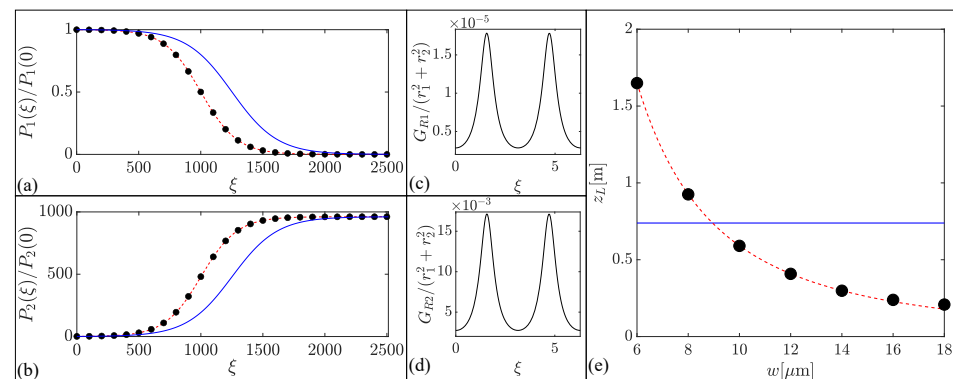


Figure 7. Changes in the powers of the pump (a) and signal (b) along the fiber's length when both beams are launched with the same width into a fiber with (red) and without self-imaging (blue). The effective loss (c,d) gain coefficients are plotted over two self-imaging periods. (e) The pump's depletion length as a function of its input width. The solid dots show numerical results for comparison.

The pump's depletion also depends on its initial width and occurs at shorter distances as the width increases. This feature is shown in part (e) of Figure 7, where the pump's depletion length is plotted as a function of the input pump width. This length is defined as the distance at which the pump's power is reduced to 1% of its initial value. The effectiveness of the variational analysis can be appreciated here. Solid dots were obtained from full numerical simulations, and each dot required 10 h of computational time on one computer. The red curve, obtained from the variational analysis, required only a few minutes of computing time on the same computer.

5. Conclusions

Graded-index (GRIN) fibers have been used in recent years for making high-power Raman amplifiers because they can improve the spatial quality of the amplified beam. We have developed a numerical model that solves the coupled pump and signal equations using both numerical and semi-analytical techniques. These equations include not only the signal's amplification and pump's depletion but also various nonlinear interactions (such as SPM and XPM) occurring between the two beams and the self-imaging effects occurring within the GRIN fiber.

We have solved the coupled nonlinear equations for the pump and signal beams numerically over a wide range of input parameters. As numerical simulations are quite time-consuming, we have also employed the variational technique to obtain simpler equations that can be solved much faster than the full model and still agree with it in most cases of practical interest. We used these equations to study how the signal's power increases and the pump's power decreases along the fiber's length and how these powers depend on various input parameters. We solved variational equations over a wide range of input parameters and found them to agree well, in most cases, with the results obtained numerically.

Our focus in this study was on the impact of self-imaging, which leads to periodic focusing and defocusing of the pump and signal beams on a millimeter scale. As expected, self-imaging leads to such oscillations in all beam parameters, a feature of GRIN fibers that masks the long-range evolution of these parameters. We filtered out the self-imaging oscillations by averaging over one self-imaging period ($\xi_p = \pi$) and used the average value of the beam parameters to optimize the performance of a GRIN fiber Raman amplifier.

Several features are worth mentioning here. First, we found that the XPM-induced coupling between the pump and signal beams introduces oscillations in their widths at a scale of 0.1 m that affect the amplification process considerably. Second, the periodic focusing and defocusing of the pump beam enhances the amplifier's gain by a large factor in each focusing region. This results in a step-like increase in the signal's power. Third, self-imaging also enhances the overall gain of the amplifier. As a result, GRIN fibers provide larger output powers compared to step-index fibers for a given set of input parameters. As GRIN fibers also improve the spatial quality of the amplified beam, their use makes for better Raman amplifiers. The results of this study should be useful for optimizing the design of such amplifiers.

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Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of Interest.

Appendix A. List of Parameters Used for Figure 4

For all three cases, the pump power, signal power, and signal widths were fixed. This corresponds to the terms in Table A1, with the terms in Table A2 being the terms that change with the changing conditions. All the terms without dimensions are the normalized coefficients derived from the normalization procedure in Equation (10).

Table A1. Fixed parameters and coefficients.

Parameters	Symbols	Values [Units]
Initial Signal Width	w_{20}	14.3 [μm]
Initial Pump Power	P_{10}	100 [kW]
Initial Signal Power	P_{20}	1 [kW]
Initial Signal Intensity	I_{20}	1.55×10^{12} [W/m^2]
Signal SPM Coefficient	γ_2	8.8×10^{-5}
Pump XPM Coefficient	Γ_1	9.1×10^{-5}
Pump Raman Coefficient	G_{R1}	5.7×10^{-5}

Table A2. Case-wise parameters and coefficients.

Parameters	Symbols	Values [Units] ($r = 0.67$)	Values [Units] ($r = 1$)	Values [Units] ($r = 1.33$)
Initial Pump Width	w_{10}	9.6 [μm]	14.3 [μm]	19.1 [μm]
Initial Pump Intensity	I_{10}	3.44×10^{14} [W/m^2]	1.55×10^{14} [W/m^2]	8.75×10^{13} [W/m^2]
Signal Raman coefficient	G_{R2}	0.0122	0.0055	0.0031
Signal XPM coefficient	Γ_2	0.0195	0.0088	0.0050
Pump SPM coefficient	γ_1	0.0203	0.0091	0.0052

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